

NORMABILITY OF PROBABILISTIC NORMED SPACES

BERNARDO LAFUERZA-GUILLÉN, JOSÉ ANTONIO RODRÍGUEZ-LALLENA,
AND CARLO SEMPI

ABSTRACT. Relying on Kolmogorov's classical characterization of normable Topological Vector spaces, we study the normability of those Probabilistic Normed Spaces that are also Topological Vector spaces and provide a characterization of normable Šerstnev spaces. We also study the normability of other two classes of Probabilistic Normed Spaces.

1. INTRODUCTION

Probabilistic Normed spaces were introduced by Šerstnev in [13]; their definition was generalized in [1], a paper that revived the study of these spaces. We recall the definition, the properties and the examples of Probabilistic Normed spaces that will be used in the following.

Let Δ be the space of distribution functions and $\Delta^+ := \{F \in \Delta : F(0) = 0\}$ the subset of distance distribution functions [11]. The space Δ can be metrized in several equivalent ways [14, 10, 12, 15] in such a manner that the metric topology coincides with the topology of weak convergence for distribution functions. Here, we assume that Δ is metrized by the Sibley metric d_S , which is the same metric denoted by d_L in [11]. We shall also consider the subset $\mathcal{D}^+ \subset \Delta^+$ of the proper distance distribution functions, i.e. those $F \in \Delta^+$ for which $\lim_{x \rightarrow +\infty} F(x) = 1$.

A *triangle function* is a mapping $\tau : \Delta^+ \times \Delta^+ \rightarrow \Delta^+$ that is commutative, associative, nondecreasing in each variable and has ϵ_0 as identity, where ϵ_a ($a \leq +\infty$) is the distribution function defined by

$$\epsilon_a(t) := \begin{cases} 0, & t \leq a, \\ 1, & t > a. \end{cases}$$

Given a nonempty set S , a mapping \mathcal{F} from $S \times S$ into Δ^+ and a triangle function τ , a *Probabilistic Metric Space* (briefly a PM space) is the triple (S, \mathcal{F}, τ) with the following properties, where we set $F_{p,q} := \mathcal{F}(p, q)$,

- (M1) $F_{p,q} = \epsilon_0$ if, and only if, $p = q$;
- (M2) $F_{p,q} = F_{q,p}$ for all p and q in S ;
- (M3) $F_{p,r} \geq \tau(F_{p,q}, F_{q,r})$ for all $p, q, r \in S$.

A *Probabilistic Normed Space* (briefly a PN space) is a quadruple (V, ν, τ, τ^*) , where V is a vector space, τ and τ^* are continuous triangle functions such that $\tau \leq \tau^*$ and ν is a mapping from V into Δ^+ , called the *probabilistic norm*, such that for every choice of p and q in V the following conditions hold:

- (N1) $\nu_p = \epsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
- (N2) $\nu_{-p} = \nu_p$;
- (N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
- (N4) $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0, 1]$.

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The pair (V, ν) is called a *Probabilistic Seminormed space* (PSN space for short) if ν satisfies (N1) and (N2).

There are special PN spaces, only some of which we list below; for the others we refer to [7].

When there is a continuous t -norm T (see [11, 3]) such that $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, where $T^*(x, y) := 1 - T(1 - x, 1 - y)$,

$$\tau_T(F, G)(x) := \sup_{s+t=x} T(F(s), G(t)) \quad \text{and} \quad \tau_{T^*}(F, G)(x) := \inf_{s+t=x} T^*(F(s), G(t))$$

the PN space $(V, \nu, \tau_T, \tau_{T^*})$ is called a *Menger PN space*, and is denoted by (V, ν, T) .

A PN space is called a *Šerstnev space* if it satisfies (N1), (N3) and the following condition, which implies both (N2) and (N4)

$$\forall p \in V \quad \forall \alpha \in \mathbf{R} \setminus \{0\} \quad \forall x > 0 \quad \nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right).$$

One speaks of an *equilateral* PN space when there is $F \in \Delta^+$ different from both ϵ_0 and ϵ_∞ such that, for every $p \neq \theta$, $\nu_p = F$, and when $\tau = \tau^* = \mathbf{M}$, which is the triangle function defined for G and H in Δ^+ by $\mathbf{M}(G, H)(x) := \min\{G(x), H(x)\}$.

Let $G \in \Delta^+$ be different from ϵ_0 and from ϵ_∞ and let $(V, \|\cdot\|)$ be a normed space; then define, for $p \neq \theta$,

$$\nu_p(x) := G\left(\frac{x}{\|p\|}\right).$$

Then (V, ν, M) is a Menger space denoted by $(V, \|\cdot\|, G, M)$ ($M(x, y) := \min\{x, y\}$). In the same conditions, if ν is defined by

$$\nu_p(x) := G\left(\frac{x}{\|p\|^\alpha}\right),$$

with $\alpha \geq 0$, then the pair (V, ν) is a PSN space called α -*simple* and it is denoted by $(V, \|\cdot\|, G; \alpha)$. The α -simple spaces can be endowed with a structure of PN space in a very general setting (when $\alpha > 1$, G should be a continuous and strictly increasing function in \mathcal{D}^+ : see [7]).

See [6, 7, 8] for properties of PN spaces.

If (V, ν, τ, τ^*) is a PN space, a mapping $\mathcal{F}: V \times V \rightarrow \Delta^+$ can be defined through

$$(1) \quad \mathcal{F}(p, q) := \nu_{p-q}.$$

This function \mathcal{F} makes (V, \mathcal{F}, τ) a Probabilistic Metric Space. Every PM space can be endowed with the strong topology; this topology is generated by the *strong neighbourhoods*, which are defined as follows: for every $t > 0$, the neighbourhood $N_p(t)$ at a point p of V is defined by

$$N_p(t) := \{q \in V : d_S(\nu_{p-q}, \epsilon_0) < t\} = \{q \in V : \nu_{p-q}(t) > 1 - t\}.$$

It is known (see [11]) that (V, \mathcal{F}, τ) , where \mathcal{F} is defined by (1), and therefore (V, ν, τ, τ^*) , is a Hausdorff space, and hence, a T_1 space; moreover, it is metrizable.

2. PN SPACES AND TV SPACES

A result from [2] can be rephrased for the purpose of the present paper in the following form

Theorem 1 (Alsina, Schweizer, Sklar). *Every PN space (V, ν, τ, τ^*) , when it is endowed with the strong topology induced by the probabilistic norm ν , is a topological vector space if, and only if, for every $p \in V$ the map from \mathbf{R} into V defined by*

$$(2) \quad \alpha \mapsto \alpha p$$

is continuous.

It was proved in [2, Theorem 4] that, if the triangle function τ^* is Archimedean, i.e. if τ^* admits no idempotents other than ϵ_0 and ϵ_∞ ([11]), then the mapping (2) is continuous and, as a consequence, the PN space (V, ν, τ, τ^*) is a Topological Vector space (=TV space).

Theorem 2. (a) No equilateral space (V, F, \mathbf{M}) is a TV space.

(b) A Šerstnev space (V, ν, τ) is a TV space if, and only if, the probabilistic norm ν maps V into \mathcal{D}^+ rather than into Δ^+ , viz. $\nu(V) \subseteq \mathcal{D}^+$.

(c) If G is a continuous and strictly increasing function in \mathcal{D}^+ , then the α -simple space $(V, \|\cdot\|, G; \alpha)$ is a TV space.

Proof. Let θ denote the null vector of the linear space V . Since any PM space and, hence, any PN space, can be metrized, one can limit oneself to investigate the behaviour of sequences. Moreover, because of the linear structure of V , one can take $p \neq \theta$ and an arbitrary sequence $\{\alpha_n\}$ with $\alpha_n \neq 0$ ($n \in \mathbf{N}$) such that $\alpha_n \rightarrow 0$ as n tends to $+\infty$.

(a) For every $n \in \mathbf{N}$, one has $\nu_{\alpha_n p} = F$ while $\nu_\theta = \epsilon_0$. Therefore the map (2) is not continuous.

(b) If ν maps V into \mathcal{D}^+ , then, for every $t > 0$, one has

$$\nu_{\alpha_n p}(t) = \nu_p\left(\frac{t}{|\alpha_n|}\right) \xrightarrow{n \rightarrow +\infty} 1,$$

whence the assertion. Conversely, if there exists at least one $p \in V$ such that $\nu_p \in \Delta^+ \setminus \mathcal{D}^+$, namely such that $\nu_p(x) \xrightarrow{x \rightarrow +\infty} \beta < 1$, then

$$\nu_{\alpha_n p}(x) = \nu_p\left(\frac{x}{|\alpha_n|}\right) \xrightarrow{n \rightarrow +\infty} \beta < 1,$$

so that the mapping $\alpha \mapsto \alpha p$ is not continuous.

(c) Let $\{\alpha_n\}$ be a sequence of real numbers that goes to 0. Then, for all $p \in V$ and $x > 0$, one has

$$\nu_{\alpha_n p}(x) = G\left(\frac{x}{\|\alpha_n p\|^\alpha}\right) = G\left(\frac{x}{|\alpha_n|^\alpha \|p\|^\alpha}\right).$$

And $\lim_{n \rightarrow +\infty} \nu_{\alpha_n p}(x) = 1$ since G belongs to \mathcal{D}^+ . \square

Corollary 1. (a) A simple space $(V, \|\cdot\|, G, M)$ is a TV space if, and only if, G belongs to \mathcal{D}^+ .

(b) An EN space (S, ν) is a TV space if, and only if, ν maps S into \mathcal{D}^+ , i.e. $\nu_p \in \mathcal{D}^+$ for every $p \in S$.

Proof. Both simple spaces and EN spaces are Šerstnev spaces (see [7]). \square

If $\nu_p(x)$ is viewed as the probability $P(\|p\| < x)$ that the usual norm of p is less than x , then, the fact that, for some $p \in V$, ν_p does not belong to \mathcal{D}^+ means that $P(\|p\| < +\infty) < 1$; this is to be regarded as being “odd”. Therefore we shall call *characteristic* any PN space (V, ν, τ, τ^*) such that $\nu(V) \subseteq \mathcal{D}^+$, or, equivalently, such that ν_p belongs to \mathcal{D}^+ for every $p \in V$. Thus Theorem 2 (b) and (c) can be rephrased as follows.

Theorem 3. (a) A Šerstnev space (V, ν, τ) is a TV space if, and only if, it is characteristic.

(b) Let G be a continuous and strictly increasing distribution function in Δ^+ . Then, the α -simple space $(V, \|\cdot\|, G; \alpha)$ is a TV space if, and only if, it is characteristic.

However, in general PN spaces, the condition $\nu(V) \subseteq \mathcal{D}^+$ is not necessary to obtain a TV space: see Theorem 9 below.

3. NORMABILITY OF PN SPACES

If (V, ν, τ, τ^*) is a TV space, the question naturally arises of whether it is also normable; in other words, whether there is a norm on V that generates the strong topology. This question had been broached by Prochaska ([9]) in the case of Šerstnev PN spaces. For this case, we shall provide a complete characterization of those characteristic Šerstnev PN spaces that are indeed normable (see Theorem 6 further on). In the process, we shall need Kolmogorov's classical characterization of normability for T_1 spaces ([4]).

Theorem 4 (Kolmogorov). *A T_1 TV space is normable if, and only if, there is a neighbourhood of the origin θ that is convex and topologically bounded.*

Here, we have called *topologically bounded* a set A in a TV space E when, for every sequence $\{\alpha_n\}$ of real numbers that converges to 0 as n tends to $+\infty$ and for every sequence $\{p_n\}$ of elements of A , one has $\alpha_n p_n \rightarrow \theta$ in the topology of E .

We recall that the *probabilistic radius* of a set A in a PN space (V, ν, τ, τ^*) is the distance distribution function R_A given by $R_A(x) := \Phi_A(x-) (= \lim_{u \rightarrow x-} \Phi_A(u))$ for all $x \in]0, \infty[$, where $\Phi_A(u) := \inf\{\nu_p(u) : p \in A\}$ for all $u \in]0, \infty[$ (see [8]). A subset A of a PN space (V, ν, τ, τ^*) is said to be \mathcal{D} -bounded if, and only if, there exists a distribution function $G \in \mathcal{D}^+$ such that $\nu_p \geq G$ for every $p \in A$: in fact, we can take $G = R_A$.

3.1. The case of Šerstnev spaces. In characterizing normable Šerstnev spaces we shall need the following result.

Theorem 5. *In a characteristic Šerstnev space (V, ν, τ) the following statements are equivalent for a subset A of V :*

- (a) *A is \mathcal{D} -bounded;*
- (b) *A is topologically bounded.*

Proof. (a) \implies (b) Let A any \mathcal{D} -bounded subset of V and let $\{p_n\}$ be any sequence of elements of A and $\{\alpha_n\}$ any sequence of real numbers that converges to 0; there is no loss of generality in assuming $\alpha_n \neq 0$ for every $n \in \mathbf{N}$. Then, for every $x > 0$, and for every $n \in \mathbf{N}$,

$$\nu_{\alpha_n p_n}(x) = \nu_{p_n}\left(\frac{x}{|\alpha_n|}\right) \geq R_A\left(\frac{x}{|\alpha_n|}\right) \xrightarrow{n \rightarrow +\infty} 1.$$

Thus $\alpha_n p_n \rightarrow \theta$ in the strong topology and A is topologically bounded.

(b) \implies (a) Let A be a subset of V which is not \mathcal{D} -bounded. Then

$$R_A(x) \xrightarrow{x \rightarrow +\infty} \gamma < 1.$$

By definition of R_A , for every $n \in \mathbf{N}$ there is $p_n \in A$ such that

$$\nu_{p_n}(n^2) < \frac{1+\gamma}{2} < 1.$$

If $\alpha_n = 1/n$, then

$$\nu_{\alpha_n p_n}(1/2) \leq \nu_{\alpha_n p_n}(n) = \nu_{p_n}(n^2) < \frac{1+\gamma}{2} < 1,$$

which shows that $\{\nu_{\alpha_n p_n}\}$ does not tend to ϵ_0 , even if it has a weak limit, viz. $\{\alpha_n p_n\}$ does not tend to θ in the strong topology; in other words, A is not topologically bounded. \square

As a consequence of the previous results, it is now possible to characterize normability for characteristic Šerstnev spaces according to the following criterion.

Theorem 6. *A characteristic Šerstnev space (V, ν, τ) is normable if, and only if, the null vector θ has a convex \mathcal{D} -bounded neighbourhood.*

The following (restrictive) sufficient condition is in [9]; we prove it here not only for the sake of completeness, but also because Prochaska's thesis is not easily accessible and, moreover, because the notation it adopts is different from the one that has become usual after the publication of [11].

Theorem 7 (Prochaska). *A characteristic Šerstnev space (V, ν, τ) with $\tau = \tau_M$ is locally convex.*

Proof. It suffices to consider the family of neighbourhoods of the origin θ , $N_\theta(t)$, with $t > 0$. Let $t > 0$, $p, q \in N_\theta(t)$ and $\alpha \in [0, 1]$. Then

$$\begin{aligned} \nu_{\alpha p + (1-\alpha)q}(t) &\geq \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)q})(t) \\ &= \sup_{\beta \in [0,1]} M(\nu_{\alpha p}(\beta t), \nu_{(1-\alpha)q}((1-\beta)t)) \\ &\geq M(\nu_{\alpha p}(\alpha t), \nu_{(1-\alpha)q}((1-\alpha)t)) = M(\nu_p(t), \nu_q(t)) > 1 - t. \end{aligned}$$

Thus $\alpha p + (1 - \alpha)q$ belongs to $N_\theta(t)$ for every $\alpha \in [0, 1]$. \square

It is well known that every simple PN space $(V, \|\cdot\|, G, M)$ with $G \in \mathcal{D}^+$ satisfies the assumptions of Theorem 7. Moreover, these PN spaces are trivially normable, since their strong topology coincides with the topology of their classical norm. In general, it is expected that most of the PN spaces considered in Theorem 7 will be normable, as shown by the following corollary.

Corollary 2. *Let (V, ν, τ_M) be a characteristic Šerstnev space. If $N_\theta(t)$ is \mathcal{D} -bounded for some $t \in]0, 1[$, then (V, ν, τ_M) is normable.*

3.2. Other cases. Apart from the Šerstnev spaces, we can also determine whether an α -simple space is normable, as the following result shows.

Theorem 8. *Let G be a continuous and strictly increasing distribution function in \mathcal{D}^+ . Then, the α -simple space $(V, \|\cdot\|, G; \alpha)$ is normable.*

Proof. Let $N_\theta(t)$ be a neighbourhood of the origin θ with $t \in]0, 1[$; then

$$N_\theta(t) = \left\{ p \in V : G\left(\frac{t}{\|p\|^\alpha}\right) > 1 - t \right\} = \left\{ p \in V : \|p\| < \left(\frac{t}{G^{-1}(1-t)}\right)^{1/\alpha} \right\}.$$

Since $h(t) = (t/G^{-1}(1-t))^{1/\alpha}$ is a continuous function such that $\lim_{t \rightarrow 0+} h(t) = 0$ and $\lim_{t \rightarrow 1-} h(t) = \infty$, then it is clear that the strong topology for V coincides with the topology of the norm $\|\cdot\|$ in V . Therefore, $(V, \|\cdot\|, G; \alpha)$ is normable. \square

It is natural to ask whether results similar to that of Theorem 6 hold for general PN spaces. The conditions of Theorem 5 need not be equivalent; for, there are PN spaces in which a set A may be topologically bounded without being \mathcal{D} -bounded. On the other hand, even in those cases, sometimes it is possible to establish directly whether a PN space that is also a TV space is normable. To illustrate both facts, we next introduce a new class of PN spaces whose interest exceeds to serve as an example at this point. Recall that only a few types of PN spaces are known: finding a new type might be useful to deep into the subject.

Before introducing the new class of PN spaces we need the following technical lemma.

Lemma 1. *Let $f: [0, +\infty[\rightarrow [0, 1]$ be a right-continuous nonincreasing function. Let us define $f^{[-1]}(1) := 0$ and $f^{[-1]}(y) := \sup\{x : f(x) > y\}$ for all $y \in [0, 1[$ ($f^{[-1]}(y)$ might be infinite). If $x_0 \in [0, +\infty[$ and $y_0 \in [0, 1]$, then the following facts are equivalent: (a) $f(x_0) > y_0$; (b) $x_0 < f^{[-1]}(y_0)$.*

Proof. If $f(x_0) > y_0$ then $f^{[-1]}(y_0) = \sup\{x : f(x) > y_0\} \geq x_0$. If we suppose that $\sup\{x : f(x) > y_0\} = x_0$, then $f(x) \leq y_0$ for every $x > x_0$. Thus $f(x_0) = f(x_0+) \leq y_0$, against the hypothesis; whence (a) \Rightarrow (b). The converse result is an immediate consequence of the nonincreasingness of f . \square

The following theorem introduces a new class of PN spaces—which generalizes an example in [5]—, and also provides some properties of the spaces in that class. As it has been said above, such properties are interesting in order to our purposes in this paper.

Theorem 9. *Let $(V, \|\cdot\|)$ a normed space and let T be a continuous t -norm. Let f be a function as in Lemma 1, and satisfying the following two properties:*

- (a) $f(x) = 1$ if and only if $x = 0$;
- (b) $f(\|p + q\|) \geq T(f(\|p\|), f(\|q\|))$ for every $p, q \in V$.

If $\nu: V \rightarrow \Delta^+$ is given by

$$(3) \quad \nu_p(x) = \begin{cases} 0, & x \leq 0, \\ f(\|p\|), & x \in]0, +\infty[, \\ 1, & x = +\infty, \end{cases}$$

for every $p \in V$, then $(V, \nu, \tau_T, \tau_{T^})$ is a Menger PN space satisfying the following properties:*

- (F1) $(V, \nu, \tau_T, \tau_{T^*})$ is a TV space;
- (F2) $(V, \nu, \tau_T, \tau_{T^*})$ is normable;
- (F3) If $p \in V$ and $t > 0$, then the strong neighbourhood $N_p(t)$ in $(V, \nu, \tau_T, \tau_{T^*})$ is not \mathcal{D} -bounded, but $N_p(t)$ is topologically bounded whenever $N_p(t) \neq V$;
- (F4) $(V, \nu, \tau_T, \tau_{T^*})$ is not a Šerstev space;
- (F5) $(V, \nu, \tau_T, \tau_{T^*})$ is not a characteristic PN space.

Proof. First, we prove that $(V, \nu, \tau_T, \tau_{T^*})$ satisfied the four axioms to be a Menger PN space:

(N1) $\nu_p = \epsilon_0 \Leftrightarrow f(\|p\|) = 1 \Leftrightarrow \|p\| = 0 \Leftrightarrow p = \theta$.

(N2) Trivial.

(N3) Given $p, q \in V$, then we have $\nu_{p+q} \geq \tau_T(\nu_p, \nu_q) \Leftrightarrow \nu_{p+q}(x) \geq \tau_T(\nu_p, \nu_q)(x) = \sup_{s+t=x} T(\nu_p(s), \nu_q(t))$ for all $x \in]0, +\infty[\Leftrightarrow f(\|p + q\|) \geq T(f(\|p\|), f(\|q\|))$, as hypothesized.

(N4) Let $p \in V$ and let $\lambda \in [0, 1]$. Then, $\nu_p \leq \tau_{T^*}(\nu_{\lambda p}, \nu_{(1-\lambda)p}) \Leftrightarrow \nu_p(x) \leq \tau_{T^*}(\nu_{\lambda p}, \nu_{(1-\lambda)p})(x) = \inf_{s+t=x} T^*(\nu_{\lambda p}(s), \nu_{(1-\lambda)p}(t)) = \inf_{s+t=x} 1 - T(1 - \nu_{\lambda p}(s), 1 - \nu_{(1-\lambda)p}(t)) = 1 - \sup_{s+t=x} T(1 - \nu_{\lambda p}(s), 1 - \nu_{(1-\lambda)p}(t))$ for all $x \in]0, +\infty[\Leftrightarrow f(\|p\|) \leq 1 - \max\{1 - f(\lambda\|p\|), 1 - f((1-\lambda)\|p\|)\} = \min\{f(\lambda\|p\|), f((1-\lambda)\|p\|)\}$. Therefore, for any $p \in V$, $\nu_p \leq \tau_{T^*}(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for all $\lambda \in [0, 1] \Leftrightarrow f(\|p\|) \leq f(\alpha\|p\|)$ for all $\alpha \in [0, 1]$. Hence, $\nu_p \leq \tau_{T^*}(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for all $\lambda \in [0, 1]$ and for all $p \in V \Leftrightarrow f$ is nonincreasing.

Now we prove the four properties:

(F1) Let $p \in V$. We have to prove that the map from \mathbf{R} into V defined by $\alpha \mapsto \alpha p$ is continuous at any $\alpha \in \mathbf{R}$. Let $\gamma > 0$ (we will suppose, without loss of generality, that $\gamma \leq 1$). We must prove that there exists a real number $\delta > 0$ so that

$d_S(\nu_{\alpha'p-\alpha p}, \epsilon_0) < \gamma$ whenever $|\alpha' - \alpha| < \delta$; or, equivalently, such that $d_S(\nu_{\beta p}, \epsilon_0) < \gamma$ whenever $|\beta| < \delta$. Since $d_S(\nu_q, \epsilon_0) = \inf\{h : \nu_q(h+) > 1 - h\} = 1 - f(\|q\|)$ (here $\nu_q(h+)$ represents the limit $\lim_{u \rightarrow h+} \nu_q(u)$), then $d_S(\nu_{\beta p}, \epsilon_0) < \gamma \Leftrightarrow 1 - f(|\beta|\|p\|) < \gamma \Leftrightarrow f(|\beta|\|p\|) > 1 - \gamma \Leftrightarrow$ (Lemma 1) $|\beta|\|p\| < f^{[-1]}(1 - \gamma) \Leftrightarrow |\beta| < f^{[-1]}(1 - \gamma)/\|p\| = \delta$.

(F2) Let $p \in V$. Let $t > 0$ (we will suppose, without loss of generality, that $t < 1 - \lim_{x \rightarrow \infty} f(x)$). Then, $N_p(t) = \{q \in V : d_S(\nu_{p-q}, \epsilon_0) < t\} = \{q \in V : 1 - f(\|p - q\|) < t\} = \{q \in V : f(\|p - q\|) > 1 - t\} =$ (Lemma 1) $\{q \in V : \|p - q\| < f^{[-1]}(1 - t)\} = B(p, f^{[-1]}(1 - t))$, i.e., the strong neighbourhood $N_p(t)$ is a ball in $(V, \|\cdot\|)$ with center in p . Conversely, let $r > 0$. If $t = 1 - f(r)$, then $f^{[-1]}(1 - t) < r$, whence $N_p(t) = B(p, f^{[-1]}(1 - t)) \subset B(p, r)$. Therefore, the strong topology for $(V, \nu, \tau_T, \tau_{T^*})$ coincides with the topology of the norm in $(V, \|\cdot\|)$.

(F3) If $p \in V$ and $0 < t < 1 - \lim_{x \rightarrow \infty} f(x)$, then $N_p(t) = B(p, f^{[-1]}(1 - t))$ is a ball in $(V, \|\cdot\|)$, whence $N_p(t)$ is topologically bounded. On the other hand, if $0 < x < \infty$ then $\Phi_{N_p(t)}(x) = \inf\{\nu_q(x) : q \in N_p(t)\} = \inf\{f(\|q\|) : \|p - q\| < f^{[-1]}(1 - t)\} = f(\|p\| + f^{[-1]}(1 - t))$. Thus, $\lim_{x \rightarrow \infty} R_{N_p(t)}(x) = f(\|p\| + f^{[-1]}(1 - t)) < 1$, i.e., $N_p(t)$ is not \mathcal{D} -bounded.

(F4) It is immediate to check that $(V, \nu, \tau_T, \tau_{T^*})$ is a Šerstnev space if and only if the function f is constant on $]0, \infty[$. From hypothesis (a) this constant should be less than 1, which contradicts the right-continuity of f at $x = 0$. Thus, $(V, \nu, \tau_T, \tau_{T^*})$ is not a Šerstnev space.

(F5) It is immediate that $\nu(V \setminus \{\theta\}) \subseteq \Delta^+ \setminus \mathcal{D}^+$. \square

Observe that the proof of (N4) is independent of the t -norm T . Thus, we can also take $\tau^* = \tau_{M^*} = \tau_M$ (see [11]) in the PN-space of Theorem 9.

Now we consider some particular cases and provide some examples which apply the preceding theorem.

Example 1. Suppose that $T = \Pi$ in Theorem 9. In this case the property (b) is read as $f(\|p + q\|) \geq f(\|p\|)f(\|q\|)$ for all $p, q \in V$. It is not difficult to prove that, under the established hypotheses for f , the property (b) is equivalent to the following one:

$$(4) \quad f(x + y) \geq f(x)f(y) \text{ for all } x, y \in [0, \infty[.$$

It is easy to check that instances of functions f satisfying the hypotheses of Theorem 9 for this case are

$$\begin{aligned} f_{\alpha, \beta}(x) &:= 1 - \frac{\beta}{\alpha} + \frac{\beta}{x + \alpha}, \quad 0 \leq \beta \leq \alpha, \\ g_{\alpha, \beta}(x) &:= 1 - \alpha + \alpha \exp(-x^\beta), \quad 0 < \alpha \leq 1, \beta > 0. \end{aligned}$$

Example 2. Suppose that $T = W$ in Theorem 9. In this case the property (b) is read as $f(\|p + q\|) \geq f(\|p\|) + f(\|q\|) - 1$ for all $p, q \in V$. Since W is the minimum continuous t -norm, all the functions f satisfying the hypotheses of Theorem 9 with respect to any t -norm T also satisfy such hypotheses with respect to W . It is not difficult to prove that, under those hypotheses, the property (b) is equivalent to the following one:

$$1 + f(x + y) \geq f(x) + f(y) \text{ for all } x, y \in [0, \infty[.$$

Instances of functions f satisfying these hypotheses but not the ones considered in Example 1—since they do not satisfy (4)—are

$$h_{\alpha, \beta}(x) := \begin{cases} 1 - \alpha x, & 0 \leq x \leq \beta, \\ 1 - \alpha \beta, & x > \beta, \end{cases} \quad 0 < \beta \leq 1/\alpha.$$

4. CONCLUSION

In what precedes we have been able to characterize those Šerstnev spaces that are normable TV spaces. Several questions remain open: to give at least sufficient conditions under which a general PN space is normable; more, to characterize (rather than just having a sufficient condition) the class of PN spaces that are also TV spaces, and, once this has been achieved, to study normability in the class thus determined.

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DEPARTAMENTO DE ESTADÍSTICA Y MATEMÁTICA APLICADA, UNIVERSIDAD DE ALMERÍA, 04120 ALMERÍA, SPAIN

E-mail address: blafuerz@ual.es

E-mail address: jarodrig@ual.es

DIPARTIMENTO DI MATEMATICA “ENNIO DE GIORGI”, UNIVERSITÀ DI LECCE, 73100 LECCE, ITALY

E-mail address: carlo.sempi@unile.it